Exercise 6

The table gives the population of Indonesia, in millions, for the second half of the 20th century.

Year	Population
1950	83
1960	100
1970	122
1980	150
1990	182
2000	214

(a) Assuming the population grows at a rate proportional to its size, use the census figures for 1950 and 1960 to predict the population in 1980. Compare with the actual figure.

(b) Use the census figures for 1960 and 1980 to predict the population in 2000. Compare with the actual population.

(c) Use the census figures for 1980 and 2000 to predict the population in 2010 and compare with the actual population of 243 million.

(d) Use the model in part (c) to predict the population in 2020. Do you think the prediction will be too high or too low? Why?

Solution

Part (a)

Use an exponential model for the population.

$$P(t) = P_0 e^{kt}$$

Use the populations at 1950 and 1960 to construct a system of equations for the two unknowns, P_0 and k.

$$\begin{cases} P(1950) = P_0 e^{k(1950)} = 83 \\ P(1960) = P_0 e^{k(1960)} = 100 \end{cases}$$

Divide both sides of the second equation by those of the first equation to eliminate P_0.

$$\frac{P_0 e^{k(1960)}}{P_0 e^{k(1950)}} = \frac{100}{83}$$

$$e^{10k} = \frac{100}{83}$$

$$\ln e^{10k} = \ln \frac{100}{83}$$

$$10k = \ln \frac{100}{83}$$

$$k = \frac{1}{10} \ln \frac{100}{83} \approx 0.018633 \text{ year}^{-1}$$

Substitute this formula for k into either of the two equations to get P_0.

$$P_0 e^{k(1950)} = 83$$

$$P_0 e^{\left(\frac{1}{10} \ln \frac{100}{83}\right)(1950)} = 83$$

$$P_0 = \frac{83}{e^{\left(\frac{1}{10} \ln \frac{100}{83}\right)(1950)}} \approx 1.37818 \times 10^{-14} \text{ million}$$

Therefore, the population model using the populations at 1950 and 1960 is

$$P(t) = \left[\frac{83}{e^{\left(\frac{1}{10} \ln \frac{100}{83}\right)(1950)}}\right] e^{\left(\frac{1}{10} \ln \frac{100}{83}\right)t}.$$

The Indonesia population in 1980 is

$$P(1980) = \left[\frac{83}{e^{\left(\frac{1}{10} \ln \frac{100}{83}\right)(1950)}}\right] e^{\left(\frac{1}{10} \ln \frac{100}{83}\right)1980} \approx 145.159 \text{ million.}$$

Use the percent difference formula to see how far off this number is from the actual value.

$$\frac{145.159 - 150}{150} \times 100\% \approx -3.22737\%$$

The model's population in 1980 underestimates the actual value by about 3.23%.

Part (b)

Use an exponential model for the population.

$$P(t) = P_0 e^{kt}$$

Use the populations at 1960 and 1980 to construct a system of equations for the two unknowns, P_0 and k.

$$\begin{cases} P(1960) = P_0 e^{k(1960)} = 100 \\ \\ P(1980) = P_0 e^{k(1980)} = 150 \end{cases}$$

Divide both sides of the second equation by those of the first equation to eliminate P_0.

$$\frac{P_0 e^{k(1980)}}{P_0 e^{k(1960)}} = \frac{150}{100}$$

$$e^{20k} = \frac{3}{2}$$

$$\ln e^{20k} = \ln \frac{3}{2}$$

$$20k = \ln \frac{3}{2}$$

$$k = \frac{1}{20} \ln \frac{3}{2} \approx 0.0202733 \text{ year}^{-1}$$

Substitute this formula for k into either of the two equations to get P_0.

$$P_0 e^{k(1960)} = 100$$

$$P_0 e^{\left(\frac{1}{20} \ln \frac{3}{2}\right)(1960)} = 100$$

$$P_0 = \frac{100}{e^{\left(\frac{1}{20} \ln \frac{3}{2}\right)(1960)}} \approx 5.53422 \times 10^{-16} \text{ million}$$

Therefore, the population model using the populations at 1960 and 1980 is

$$P(t) = \left[\frac{100}{e^{\left(\frac{1}{20} \ln \frac{3}{2}\right)(1960)}}\right] e^{\left(\frac{1}{20} \ln \frac{3}{2}\right)t}.$$

The Indonesia population in 2000 is

$$P(2000) = \left[\frac{100}{e^{\left(\frac{1}{20} \ln \frac{3}{2}\right)(1960)}}\right] e^{\left(\frac{1}{20} \ln \frac{3}{2}\right)2000} = 225 \text{ million}.$$

Use the percent difference formula to see how far off this number is from the actual value.

$$\frac{225 - 214}{214} \times 100\% \approx 5.14019\%$$

The model's population in 2000 overestimates the actual value by about 5.14%.

Part (c)

Use an exponential model for the population.

$$P(t) = P_0 e^{kt}$$

Use the populations at 1980 and 2000 to construct a system of equations for the two unknowns, P_0 and k.

$$\begin{cases} P(1980) = P_0 e^{k(1980)} = 150 \\ P(2000) = P_0 e^{k(2000)} = 214 \end{cases}$$

Divide both sides of the second equation by those of the first equation to eliminate P_0.

$$\frac{P_0 e^{k(2000)}}{P_0 e^{k(1980)}} = \frac{214}{150}$$

$$e^{20k} = \frac{107}{75}$$

$$\ln e^{20k} = \ln \frac{107}{75}$$

$$20k = \ln \frac{107}{75}$$

$$k = \frac{1}{20} \ln \frac{107}{75} \approx 0.017767 \text{ year}^{-1}$$

Substitute this formula for k into either of the two equations to get P_0.

$$P_0 e^{k(1980)} = 150$$

$$P_0 e^{\left(\frac{1}{20} \ln \frac{107}{75}\right)(1980)} = 150$$

$$P_0 = \frac{150}{e^{\left(\frac{1}{20} \ln \frac{107}{75}\right)(1980)}} \approx 7.90974 \times 10^{-14} \text{ million}$$

Therefore, the population model using the populations at 1980 and 2000 is

$$P(t) = \left[\frac{150}{e^{\left(\frac{1}{20} \ln \frac{107}{75}\right)(1980)}}\right] e^{\left(\frac{1}{20} \ln \frac{107}{75}\right)t}.$$

The Indonesia population in 2010 is

$$P(2010) = \left[\frac{150}{e^{\left(\frac{1}{20} \ln \frac{107}{75}\right)(1980)}}\right] e^{\left(\frac{1}{20} \ln \frac{107}{75}\right)2010} \approx 255.608 \text{ million}.$$

Use the percent difference formula to see how far off this number is from the actual value.

$$\frac{255.608 - 243}{243} \times 100\% \approx 5.18862\%$$

The model's population in 2010 overestimates the actual value by about 5.19%.

Part (d)

Using the model from part (c), the Indonesia population in 2020 is

$$P(2020) = \left[\frac{150}{e^{\left(\frac{1}{20}\ln\frac{107}{75}\right)(1980)}}\right] e^{\left(\frac{1}{20}\ln\frac{107}{75}\right)2020} \approx 305.307 \text{ million.}$$

This prediction is likely too high because exponential population growth is unsustainable due to limited resources and space.